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Improved estimates of rare K decay matrix-elements from $K_{\ell 3}$ decays

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Abstract

The estimation of rare K decay matrix-elements from $K_{\ell 3}$ experimental data is extended beyond LO in Chiral Perturbation Theory. Isospin-breaking effects at NLO (and partially NNLO) in the ChPT expansion, as well as QED radiative corrections, are now accounted for. The analysis relies mainly on the cleanness of two specific ratios of form-factors, for which the theoretical control is excellent. As a result, the uncertainties on the $K^+ \to \pi^+ \nu\bar{\nu}$ and $K_L \to \pi^0 \nu\bar{\nu}$ matrix-elements are reduced by a factor of about 7 and 4, respectively, and similarly for the direct CP-violating contributions to $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$. They could be reduced even further with better experimental data for the $K_{\ell 3}$ slopes and the $K_{\ell 3}^+$ branching ratios. As a result, the non-parametric errors for $\mathcal{B}(K \to \pi\nu\bar{\nu})$ and for the direct CP-violating contributions to $\mathcal{B}\left(K_L \to \pi^0 \ell^+ \ell^-\right)$ are now completely dominated by those on the short-distance physics.

1 Introduction

The rare decays $K \to \pi\nu\bar{\nu}$ and $K_L \to \pi^0\ell^+\ell^-$ are important tools to test the Standard Model (SM), and to search for possible new physics. As they proceed through flavor changing neutral currents (FCNC), though they are very suppressed in the SM, they show an exceptional sensitivity to short-distance physics. Recent theoretical progress has greatly improve the theoretical control over the SM predictions [1–3]. On the experimental side, rare K programs at CERN and J-PARC are currently under study, aiming at a measurement by the beginning of the next decade.

An essential aspect of these decay modes is of being semi-leptonic. Contrary to ε'/ε for example, the dominant contribution does not come from four-quark operators, but rather from quark vector-current FCNC operators like $(\bar{s}\gamma_\mu d)(\bar{\nu}\gamma^\mu(1-\gamma_5)\nu)$, on which we have an excellent control. This stems from their relationship under the isospin symmetry with the charge-current (CC) Fermi operators $(\bar{s}\gamma_\mu u)(\ell\gamma^\mu(1-\gamma_5)\bar{\nu}_\ell)$. More specifically, the FCNC hadronic matrix-elements required for rare K decays as well as the one for the CC transition in $K \to \pi\ell\nu_\ell$ decays ($K_{\ell 3}$) are parametrized as ($T = P - K$)

$$\langle \pi^j(K)|\bar{q}\gamma^\mu\lambda_a q|K^i(P)\rangle = C_{ij}(f_+^{K^i\pi^j}(T^2)(P+K)^\mu + f_-^{K^i\pi^j}(T^2)(P-K)^\mu)\,, \tag{1}$$

with C_{ij} some Clebsch-Gordan coefficients and the Gell-Mann matrices $\lambda_{FCNC} = \lambda_6 \pm i\lambda_7$, $\lambda_{CC} = \lambda_4 \pm i\lambda_5$ projecting out the desired quark-flavor structures. In the isospin limit, all these form-factors are equal.

At lowest order in Chiral Perturbation Theory (ChPT), these form-factors are all related to the conserved current form-factors $f_\pm^{\pi^+\pi^-}(T^2)$, and thus $f_+^{K^i\pi^j}(T^2) = 1$, $f_-^{K^i\pi^j}(T^2) = 0$ for all $i,j = +,0$. Further, the Ademollo-Gatto theorem protects against large $SU(3)$ corrections, which can arise only at the second order in $m_s - m_{u,d}$. Of course, in practice, it is not useful for us to compute these $SU(3)$ corrections since they are the same for all $K \to \pi$ transitions. Indeed, our goal is to use instead the precise experimental information on the form-factors obtained from $K_{\ell 3}$ decays. The problem then reduces to the study of isospin-breaking effects, proportional to $\varepsilon^{(2)} \sim (m_u - m_d)/m_s$, to relate FCNC and CC form-factors as precisely as possible.

This strategy of using $K_{\ell 3}$ data is common practice, but currently relies on the $\mathcal{O}(p^2\varepsilon^{(2)})$, LO analysis of Ref. [4]. Given the recent theoretical progress in the computation of short-distance QCD corrections, it is now time to improve and go beyond LO. More precisely, with respect to Ref. [4], our goals are:

1 – To include isospin-breaking effects at NLO (and partially NNLO) in the ChPT expansion.

2 – To account for QED radiative corrections, at leading order in the ChPT expansion.

3 – To update matrix-elements using the latest $K_{\ell 3}$ experimental data.

4 – To perform a detailed error study, including both theoretical and experimental uncertainties.

The outline of the paper is as follows. In the next Section, the master formulas are given. In Section 3, the ChPT results for the form-factors at $\mathcal{O}(p^4\varepsilon^{(2)})$, and partially $\mathcal{O}(p^6\varepsilon^{(2)})$, are given and discussed. Special emphasis will be set on two ratios of form-factors on which an exceptional theoretical control can be reached. Then, the numerical analysis is performed in Section 4 and our results are summarized in the Conclusion. Finally, loop functions as well as the details of the computation of the QED radiative corrections are presented in the Appendix.

2 Generalities

The FCNC weak Hamiltonian relevant for the $K \to \pi\nu\bar{\nu}$ decays and for the direct CP-violating contribution to the $K_L \to \pi^0\ell^+\ell^-$ decays is

$$H_{eff} = \frac{G_F\alpha(M_Z)}{\sqrt{2}}\sum_{\ell=e,\mu,\tau}\left(\frac{y_\nu}{2\pi\sin^2\theta_W}Q_{\nu\bar{\nu}}\right) - \frac{G_F\alpha(M_Z)}{\sqrt{2}}\sum_{\ell=e,\mu}(y_{7V}Q_{7V} + y_{7A}Q_{7A}) + \text{h.c.}\,, \tag{2}$$

$$Q_{\nu\bar{\nu}} = (\bar{s}\gamma_\mu d)\times(\bar{\nu}_\ell\gamma^\mu(1-\gamma_5)\nu_\ell)\,,\quad Q_{7V} = (\bar{s}\gamma_\mu d)\times\left(\bar{\ell}\gamma^\mu\ell\right),\quad Q_{7A} = (\bar{s}\gamma_\mu d)\times\left(\bar{\ell}\gamma^\mu\gamma_5\ell\right)\,. \tag{3}$$

In the SM, the Wilson coefficient y_ν is given by

$$y_\nu = (\mathrm{Re}\lambda_t + i\mathrm{Im}\lambda_t)\, X_t + |V_{us}|^4\, \mathrm{Re}\lambda_c P_{u,c}\ , \tag{4}$$

with $\lambda_q = V_{qs}^* V_{qd}$, $X_t = 1.464 \pm 0.041$ [1], $P_{u,c} = 0.41 \pm 0.04$ for $m_c\,(m_c) = 1.30 \pm 0.05$ [1, 2], while y_{7V} and y_{7A} are given by [5]

$$y_{7V}\,(\mu \approx 1\ \mathrm{GeV}) = (0.73 \pm 0.04)\ \mathrm{Im}\lambda_t,\ \ y_{7A}\,(M_W) = (-0.68 \pm 0.03)\ \mathrm{Im}\lambda_t\ . \tag{5}$$

In the final numerical applications, we will also use the CKM parameters as obtained in Ref. [6] (compatible with Ref. [7]), and V_{us} from Ref. [8]:

$$\mathrm{Im}\,\lambda_t = 1.313^{+0.112}_{-0.063},\ \mathrm{Re}\,\lambda_t = -3.09^{+0.13}_{-0.22},\ \mathrm{Re}\,\lambda_c = -0.22091^{+0.00092}_{-0.00093},\ |V_{us}| = 0.2245 \pm 0.0016 \tag{6}$$

All the short-distance physics is encoded in the Wilson coefficients. The remaining task for computing the branching ratios is thus to get the matrix-elements of these operators, and to carry out the phase-space integration. For later use, we collect below the parametrizations of the branching ratios in terms of the form-factors. For that, we adopt the standard $K_{\ell 3}$ parametrization in terms of the form-factor values at the origin ($T^2 = 0$) and of their derivatives there, i.e. their slopes. In this respect, remember also that $f_-^{K^i \pi^j}$ can be eliminated in favor of the scalar form-factors (matrix-elements of the scalar current $\bar{q}\lambda_a q$):

$$f_0^{K^i\pi^j}\left(T^2\right) = f_+^{K^i\pi^j}\left(T^2\right) + \frac{T^2}{M_{K^i}^2 - M_{\pi^j}^2} f_-^{K^i\pi^j}\left(T^2\right)\ , \tag{7}$$

such that only the vector form-factors at the origin are needed, $f_0^{K^i\pi^j}(0) = f_+^{K^i\pi^j}(0)$.

Decay rates

The $K_{\ell 3}$ decay rates are

$$\Gamma\left(K^i \to \pi^j \ell^+ \nu_\ell\,(\gamma)\right) = C_{ij}^2 \frac{G_F^2 S_{EW} M_{K^i}^5}{192\pi^3} \left| V_{us} \times f_+^{K^i\pi^j}(0) \right|^2 \mathcal{I}_\ell^{ij} \left(1 + 2\Delta_{\ell,EM}^{ij}\right)\ , \tag{8}$$

with the Clebsch-Gordan coefficients $C_{0+} = 1$ and $C_{+0} = 1/\sqrt{2}$. The Fermi constant G_F is fixed from μ decay as $G_F = 1.166371\,(6) \cdot 10^{-11}\ \mathrm{MeV}^{-2}$ [9]. Correspondingly, the factor $S_{EW} = 1.0232$ stands for the short distance part of the corrections to the semi-leptonic weak charged current, to leading order in $m_{Z,W} \to \infty$ [10]. The phase-space integrals $\mathcal{I}_\ell^{ij}$, which are functions of the form-factor slopes, are given below.

Adopting a similar parametrization, the $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \pi^0\nu\bar{\nu}$ branching ratios are written as

$$\mathcal{B}\left(K^+ \to \pi^+\nu\bar{\nu}\,(\gamma)\right) = \kappa_\nu^+\,(1 + \Delta_{EM}) \left|\frac{y_\nu}{|V_{us}|^5}\right|^2,\ \mathcal{B}\left(K_L \to \pi^0\nu\bar{\nu}\right) = \kappa_\nu^L \left(\frac{\mathrm{Im}\,y_\nu}{|V_{us}|^5}\right)^2\ , \tag{9}$$

$$\kappa_\nu^{+,L} = \tau_{+,L} \frac{G_F^2 M_{K^{+,0}}^5 \alpha\,(M_Z)^2}{256\pi^5 \sin^4\theta_W} |V_{us}|^8 \left| V_{us} \times f_+^{K^{+,0}\pi^{+,0}}(0) \right|^2 \mathcal{I}_\nu^{+,0}\ . \tag{10}$$

We will further use [1]

$$\alpha_{\overline{MS}}\,(M_Z)^{-1} = 127.9,\ \left(\sin^2\theta_W\right)_{\overline{MS}} = 0.231\ , \tag{11}$$

when computing $\kappa_\nu^{+,L}$. The experimental errors on $\alpha_{\overline{MS}}\,(M_Z)$ and $\left(\sin^2\theta_W\right)_{\overline{MS}}$ as well as the uncertainties related to (unknown) higher-order, purely electroweak short-distance corrections are understood to be accounted for in the Wilson coefficients [11]. Finally, the direct CP-violating contribution to the $K_L \to \pi^0\ell^+\ell^-$

decays, which arises from the Q_{7A} and Q_{7V} operators (for the general structure and a description of the indirect CP-violating and CP-conserving contributions, see for example [3]), takes the form

$$\mathcal{B}\left(K_L\to\pi^0\ell^+\ell^-\right)_{\mathrm{DCPV}}=\left|\frac{\mathrm{Im}\,y_{7V}}{|V_{us}|^5}\right|^2\kappa_\ell^V+\left|\frac{\mathrm{Im}\,y_{7A}}{|V_{us}|^5}\right|^2\kappa_\ell^A\;, \tag{12}$$

$$\kappa_\ell^i=\tau_L\frac{G_F^2M_{K^0}^5\alpha\left(M_Z\right)^2}{384\pi^3}\left|V_{us}\right|^8\left|V_{us}\times f_+^{K^0\pi^0}\left(0\right)\right|^2\mathcal{I}_\ell^i\;. \tag{13}$$

In all cases, the long-distance electromagnetic corrections are moved into the $\Delta_{\ell,EM}^{ij}$'s, which thus include both virtual photon exchanges and real photon emissions. Because of the latter, these corrections depend on the experimental requirements enforced on the kinematics of the corresponding radiative decay.

For $K_{\ell 3}$, also the local QED corrections (including the electromagnetic and semi-leptonic counterterms) and the QED corrections to the phase-space are understood in $\Delta_{\ell,EM}^{ij}$. These have been computed both restricting the $K_{\ell 3\gamma}$ phase-space to the three-body kinematics [12, 13], and for the fully inclusive case [14, 15].

For $K^+\to\pi^+\nu\bar{\nu}$, the correction factor Δ_{EM} obviously concerns the hadronic part only, and will be computed later on. Finally, for $K_L\to\pi^0\ell^+\ell^-$, long-distance QED radiative corrections concern only the lepton pair and are neglected.

Phase-space integrals

We use the quadratic and linear parametrizations for the vector and scalar form-factors, respectively:

$$f_+^{K^i\pi^j}\left(q^2\right)=f_+^{K^i\pi^j}\left(0\right)\left(1+\lambda_+^{ij\prime}\frac{q^2}{m_{\pi^\pm}^2}+\lambda_+^{ij\prime\prime}\frac{q^4}{2m_{\pi^\pm}^4}\right),\;\; f_0^{K^i\pi^j}\left(q^2\right)=f_+^{K^i\pi^j}\left(0\right)\left(1+\lambda_0^{ij}\frac{q^2}{m_{\pi^\pm}^2}\right)\;. \tag{14}$$

Then, the phase-space integrals $\mathcal{I}^{ij}$ depends only on the slopes λ^{ij}. Explicitly, the $K_{\ell 3}$ phase-space integrals are ($i,j=+,0$ or $0,+$)

$$\mathcal{I}_\ell^{ij}=\int_{r_\ell^2}^{(1-r_\pi)^2}dz\left(1-\frac{r_\ell^2}{z}\right)^2\lambda_\pi\left(\left|\frac{f_+^{K^i\pi^j}\left(z\right)}{f_+^{K^i\pi^j}\left(0\right)}\right|^2\lambda_\pi^2\left(1+\frac{r_\ell^2}{2z}\right)+\frac{3}{2}\left|\frac{f_0^{K^i\pi^j}\left(z\right)}{f_+^{K^i\pi^j}\left(0\right)}\right|^2\frac{r_\ell^2}{z}\left(1-r_\pi^2\right)^2\right)\;, \tag{15}$$

with $z\equiv q^2/M_{K^i}^2$, $r_{\ell,\pi}\equiv m_{\ell,\pi^j}/M_{K^i}$, $\lambda_\pi\equiv\lambda^{1/2}\left(1,z,r_\pi^2\right)$, while those for the rare K decays are

$$\mathcal{I}_\nu^i=\int_0^{(1-r_\pi)^2}dz\lambda_\pi^3\left|\frac{f_+^{K^i\pi^i}\left(z\right)}{f_+^{K^i\pi^i}\left(0\right)}\right|^2\quad(i=+,L), \tag{16a}$$

$$\mathcal{I}_\ell^V=\int_{4r_\ell^2}^{(1-r_\pi)^2}\lambda_\pi^3\beta_\ell\left(1+\frac{2r_\ell^2}{z}\right)\left|\frac{f_+^{K^0\pi^0}\left(z\right)}{f_+^{K^0\pi^0}\left(0\right)}\right|^2, \tag{16b}$$

$$\mathcal{I}_\ell^A=\int_{4r_\ell^2}^{(1-r_\pi)^2}\lambda_\pi\beta_\ell\left(\lambda_\pi^2\left(1-\frac{4r_\ell^2}{z}\right)\left|\frac{f_+^{K^0\pi^0}\left(z\right)}{f_+^{K^0\pi^0}\left(0\right)}\right|^2+\frac{6r_\ell^2}{z}\left(1-r_\pi^2\right)^2\left|\frac{f_0^{K^0\pi^0}\left(z\right)}{f_+^{K^0\pi^0}\left(0\right)}\right|^2\right)\;. \tag{16c}$$

with $\beta_\ell\equiv\sqrt{1-4r_\ell^2/z}$ (see Appendix A for numerical expressions).

3 Vector form-factors for $K\to\pi$ transitions

As we will see in the next section, devoted to the numerical analysis, the accuracy of $K_{\ell 3}$ data are now below the percent level. To make full use of these impressive results, the theoretical analysis has to reach a corresponding level of precision. This is not an easy task as it requires some control over the $\mathcal{O}(p^6)$

corrections. Fortunately, for the purpose of estimating FCNC form-factors from those extracted from $K_{\ell 3}$ data, two well-chosen ratios are sufficient [4]. Considering ratios has the immediate advantage that only isospin-breaking corrections, proportional to $\varepsilon^{(2)} \sim (m_u - m_d)/m_s$, have to be brought under control.

Therefore, in a first stage, we have extended the $\mathcal{O}\left(p^4\varepsilon^{(2)}; p^2\alpha\right)$ analysis of Ref. [12] to the FCNC form-factors. In a second stage, the local corrections of $\mathcal{O}(p^6\varepsilon^{(2)})$ to the ratios will be discussed. The good surprise is that the structure of the $K \to \pi$ vector transitions somehow protects our two ratios from large corrections.

3.1 The vector form-factors at $\mathcal{O}\left(p^4\varepsilon^{(2)}; p^2\alpha\right)$

For the $\langle \pi^i | V^j | K^k \rangle$ vector form-factors at $\mathcal{O}\left(p^4\varepsilon^{(2)}; p^2\alpha\right)$, one has to compute meson and photon one-loop diagrams, with vertices coming from the leading $\mathcal{O}\left(p^2\varepsilon^{(2)}\right)$ chiral Lagrangian and from the $\mathcal{O}(p^0\alpha)$ Dashen term $Z_{em}e^2F^4\left\langle U^\dagger QUQ \right\rangle$ [16]. Renormalization is carried out by adding the contributions from the $\mathcal{O}(p^4\varepsilon^{(2)})$ strong counterterms L_i [17] and from the $\mathcal{O}(p^2\alpha)$ QED counterterms K_i [18]. In all these Lagrangians, the CC and FCNC vector currents are introduced through the covariant derivative with a $\lambda_4 \pm i\lambda_5$ and $\lambda_6 \pm i\lambda_7$ flavor structure, respectively, or directly through their contributions to the field-strengths.

In our computation, we keep the vector currents as external sources, to be coupled later to the lepton pairs, so that the semi-leptonic counterterms [19] are not needed to get a UV-finite result. In that respect, the situation is different for $K_{\ell 3}$ and rare K decays. For the former, these counterterms are ultimately needed to renormalize the QED corrections where a photon is exchanged between the lepton and the meson. Altogether, these form the long-distance QED corrections and are included in Δ_{EM}. On the other hand, for rare K decays, there is no such photon exchange, and no semi-leptonic counterterm is needed. Further, it should be clear that short-distance QED corrections are to be accounted for in the Wilson coefficients of the effective operators. In particular, the S_{EW} factor does not occur for rare K decays (see also the discussion in Refs. [4, 12]).

A straightforward computation gives the following $\langle \pi^i | V^j | K^k \rangle$ matrix-elements, including terms up to $\mathcal{O}\left(p^4\varepsilon^{(2)}; p^2\alpha\right)$:

$$f_+^{K^0\pi^0}\left(q^2\right) = 1 + 3H_{K^0}^{\eta} + H_{K^0}^{\pi^0} + 2H_{K^+}^{\pi^+} - \sqrt{3}\varepsilon^{(2)}(1 + H_{K^0}^{\eta} + 3H_{K^0}^{\pi^0} + 2H_{K^+}^{\pi^+}) - \sqrt{3}\varepsilon^{(4)} , \tag{17a}$$

$$f_+^{K^+\pi^0}\left(q^2\right) = 1 + 3H_{K^+}^{\eta} + H_{K^+}^{\pi^0} + 2H_{K^0}^{\pi^+} + \sqrt{3}\varepsilon^{(2)}(1 + H_{K^+}^{\eta} + 3H_{K^+}^{\pi^0} + 2H_{K^0}^{\pi^+}) + \sqrt{3}\varepsilon^{(4)} - \frac{\alpha}{2\pi}\delta_{EM}^{K^+} , \tag{17b}$$

$$f_+^{K^0\pi^+}\left(q^2\right) = 1 + 3H_{K^+}^{\eta} + H_{K^+}^{\pi^0} + 2H_{K^0}^{\pi^+} + 2\sqrt{3}\varepsilon^{(2)}(H_{K^+}^{\pi^0} - H_{K^+}^{\eta}) - \frac{\alpha}{2\pi}\delta_{EM}^{\pi^+} , \tag{17c}$$

$$f_+^{K^+\pi^+}\left(q^2\right) = 1 + 3H_{K^0}^{\eta} + H_{K^0}^{\pi^0} + 2H_{K^+}^{\pi^+} - 2\sqrt{3}\varepsilon^{(2)}(H_{K^0}^{\pi^0} - H_{K^0}^{\eta}) - \frac{\alpha}{2\pi}J_{EM}\left(q^2, M_{K^+}^2, M_{\pi^+}^2\right) , \tag{17d}$$

with

$$\delta_{EM}^i = 1 - 16\pi^2 K_{12}^r + \log\frac{M_\gamma}{\mu} - \frac{3}{4}\log\frac{M_i}{\mu} . \tag{18}$$

The loop functions are given in the appendix. The two-point functions H_i^j involve L_9, while the three-point function J_{EM} is free of counterterms but IR-divergent. The isospin-breaking terms coming from $\pi^0 - \eta$ mixing are ($2\hat{m} = m_d + m_u$)

$$\varepsilon^{(2)} = \frac{\sqrt{3}}{4}\frac{m_d - m_u}{m_s - \hat{m}} = 0.01061 \pm 0.00083 , \tag{19}$$

and $\varepsilon^{(4)} = \varepsilon_{IB}^{(4)} + \varepsilon_{EM}^{(4)}$ with [20, 21]

$$\begin{aligned}\varepsilon_{IB}^{(4)} &= \varepsilon^{(2)}\frac{\Delta(2A_K - A_\pi + A_\eta + 2M_K^2) - 2M_\pi^2(A_K - A_\pi) - 1024\pi^2\Delta^2(3L_7^r + L_8^r)}{24\pi^2F_\pi^2(M_\eta^2 - M_\pi^2)} \\ &= \varepsilon^{(2)}\left(0.27 \pm 0.09\right) = (3 \pm 1)\cdot 10^{-3} ,\end{aligned} \tag{20}$$

$$\varepsilon_{EM}^{(4)} = \alpha\frac{-9Z_{em}(M_K^2 + A_K) + 16\pi^2((2M_K^2 + M_\pi^2)Q_1^r - 6M_\pi^2 Q_2^r)}{18\sqrt{3}\pi(M_\eta^2 - M_\pi^2)} = (1 \pm 2)\cdot 10^{-4} , \tag{21}$$

where $\Delta = M_K^2 - M_\pi^2$, $A_i = M_i^2 \log\left(M_i^2/\mu^2\right)$, $Q_1^r = 3K_4^r - 6K_3^r + 2K_5^r + 2K_6^r$, $Q_2^r = K_9^r + K_{10}^r$, and $Z_{em} = (M_{\pi^\pm}^2 - M_{\pi^0}^2)/2e^2F_\pi^2 \approx 0.8$. Numerical estimates are taken from Ref. [13] (which is based on the quark-mass analysis of Ref. [22] as well as from general dimensional analysis arguments for the K_i counterterms).

Adopting the parametrization introduced in the first section, the QED corrections δ_{EM}^i and J_{EM} are moved into the Δ_{EM} corrections (as well as the small $\varepsilon_{EM}^{(4)}$ effects), and will thus be dropped from Eqs.(17). It should be clear though that not all QED effects are moved into Δ_{EM}, since the physical charged particle masses occur inside the loop functions. In this respect, the separation into long-distance QED corrections and "purely" strong parts is somewhat ambiguous.

From the general expressions, the isospin-breaking corrections for the slopes and for the form-factors at the origin can easily be obtained. The former will be discussed later on. For the latter, setting $q^2 = 0$ in Eqs.(17), one finds

$$f_+^{K^0\pi^0}(0) = 0.9775 - \sqrt{3}(0.963\varepsilon^{(2)} + \varepsilon^{(4)}) , \quad (22a)$$

$$f_+^{K^+\pi^0}(0) = 0.9773 + \sqrt{3}(0.963\varepsilon^{(2)} + \varepsilon^{(4)}) , \quad (22b)$$

$$f_+^{K^0\pi^+}(0) = 0.9773 - 0.0250\varepsilon^{(2)} , \quad (22c)$$

$$f_+^{K^+\pi^+}(0) = 0.9775 + 0.0257\varepsilon^{(2)} , \quad (22d)$$

where M_η is set to its physical value in the loop functions (as prescribed by unitarity). Setting instead

$$M_\eta^2 = (2M_{K^0}^2 + 2M_{K^\pm}^2 + M_{\pi^0}^2 - 2M_{\pi^\pm}^2)/3 , \quad (23)$$

as prescribed at this order, shifts all the form-factors down by a common 0.0004.

Inserting $\varepsilon^{(2)}, \varepsilon^{(4)}$ as given in Eqs.(19,20) in Eqs.(22), very precise estimates can be obtained. Unfortunately, the theoretical error on the isospin-symmetric $\mathcal{O}(p^6)$ correction is significantly larger than the experimental errors on the $K_{\ell 3}$ data. Specifically, the isospin-symmetric $\mathcal{O}(p^6)$ result of Ref. [23] is, using the numerical estimates for the local terms found in Ref. [24] (see also Refs. [25, 26]):

$$f_+^{K\pi}(0)\big|_{\mathcal{O}(p^6)} = \delta_{loops} + \delta_{CT} = -0.001 \pm 0.010 , \quad (24)$$

$$\delta_{loops} = 0.0146 \pm 0.0064, \;\; \delta_{CT} = -\frac{8\Delta^2}{F_\pi^4}(C_{12}^r + C_{34}^r) = -0.016 \pm 0.008 . \quad (25)$$

One should also keep in mind that the lattice estimate [27] for $f^{K\pi}(0)\big|_{\mathcal{O}(p^6)}^{lattice} = -0.016$, and thus $\delta_{CT}^{lattice} \approx -0.031$. Further, it is only with this larger $\mathcal{O}(p^6)$ correction that V_{us}, as extracted from $K_{\ell 3}$ data, satisfies the CKM unitarity. This seems to indicate that the error on δ_{CT} is under-estimated. As said, we will circumvent these difficulties by considering only two ratios of form-factors, to which we now turn.

3.2 The ratio r

Once the long-distance QED corrections have been factorized into the Δ_{EM}'s (see Eqs.(8,9)), the form-factors satisfy

$$r_{0+} \equiv \frac{f_+^{K^+\pi^0}(q^2)}{f_+^{K^0\pi^+}(q^2)} = \frac{f_+^{K^+\pi^+}(q^2)}{f_+^{K^0\pi^0}(q^2)} = 1 + \sqrt{3}(\varepsilon^{(2)} + \varepsilon^{(4)}) . \quad (26)$$

This relation is valid up to and including $\mathcal{O}\left(p^6, p^4\varepsilon^{(2)}, p^2\alpha\right)$ terms, since the isospin-exact $\mathcal{O}(p^6)$ correction drops out in the ratio. The dominant corrections then come from $\mathcal{O}(p^6\varepsilon^{(2)}, p^4\alpha)$. Further, being momentum-independent, it tells us that at this order, the slopes of the CC form-factors are identical, as well as those for the FCNC form-factors. Since the same relation holds for scalar form-factors, we can write

$$\lambda_{+,0}^{CC} \equiv \lambda_{+,0}^{K^+\pi^0} = \lambda_{+,0}^{K^0\pi^+}, \; \lambda_{+,0}^{FCNC} \equiv \lambda_{+,0}^{K^+\pi^+} = \lambda_{+,0}^{K^0\pi^0} . \quad (27)$$

From the two equalities in Eq.(26), one can form the double ratio r which is exactly 1 at this order:

$$r \equiv \frac{f_+^{K^+\pi^0}\left(q^2\right)}{f_+^{K^0\pi^+}\left(q^2\right)}\frac{f_+^{K^0\pi^0}\left(q^2\right)}{f_+^{K^+\pi^+}\left(q^2\right)} = 1 + \mathcal{O}((\varepsilon^{(2)})^2)\ . \tag{28}$$

All the momentum-dependences cancel to first order in $\varepsilon^{(2)}$ (and no further expansion in $1/F_\pi^2$ is needed). This is a striking prediction of ChPT. It can be understood from the quark diagrams for each transition, where the u or d spectator quark plays no role at leading orders, except for $\pi^0-\eta$ mixing and of course long-distance QED corrections moved into Δ_{EM}. In that respect, notice that all the QED corrections originating from the meson masses, kept in the form-factors, do also cancel out completely.

At $\mathcal{O}(p^6\varepsilon^{(2)})$, though $\pi^0-\eta$ mixing still cancels in r, one could start to feel the effect of the spectator quarks. Naively, only non-local isospin-breaking contributions at two-loops could generate corrections (for instance "sunrise"-type graphs). Still, there is a good possibility that these effects also cancel out in r, at least to a large extent. A full two-loop computation would be required to check this conjecture, and is beyond our scope. For now, we will be satisfied by looking only at the behavior of $\mathcal{O}(p^6\varepsilon^{(2)})$ local terms for the vector form-factor (thus computing only tree-level wave-function and vertex corrections from the C_i Lagrangian of Ref. [28], in which we renormalize the C_i by F_π^{-2}). We observe that while there is a local, momentum-dependent correction to Eq.(26),

$$r_{0+} = 1 + \sqrt{3}(\varepsilon^{(2)} + \varepsilon^{(4)} + \varepsilon^{(6)}) + \frac{16\Delta\varepsilon^{(2)}}{\sqrt{3}F_\pi^4}\left(\Delta(4C_{12} - 6C_{35}) + q^2\left(2C_{12} + C_{65} + C_{90}\right)\right) + \mathcal{O}\left(\alpha\right)\ , \tag{29}$$

$$\varepsilon^{(6)} = \varepsilon^{(2)}\frac{128\Delta^2(2M_K^2 + M_\pi^2)}{F_\pi^4(M_\eta^2 - M_\pi^2)}\left(\frac{C_{14}+C_{17}}{9} + \frac{C_{18}}{3} - C_{19} - 2\frac{C_{20}+C_{31}+C_{32}+2C_{33}}{3}\right)\ , \tag{30}$$

all these terms cancel out in the double ratio r, apart from a small $\mathcal{O}\left(\alpha Z_{em}\right)$ term

$$r = 1 - \frac{64\pi\alpha Z_{em}}{F_\pi^2}\left(2\Delta C_{12} - q^2 C_{90}\right) + \mathcal{O}((\varepsilon^{(2)})^2) \tag{31}$$

where $C_{12} \sim$ a few 10^{-6} [26]. This indicates that indeed, there are strong cancellations at play in the ratio r between isospin-breaking corrections induced by the spectator quark. Therefore, in our numerical applications, we will take

$$r = 1.0000 \pm 0.0002\ , \tag{32}$$

to account for possible $(\varepsilon^{(2)})^2$ effects, $\mathcal{O}(p^4\alpha)$ corrections or residual $\mathcal{O}(p^6\varepsilon^{(2)})$ non-local contributions.

3.3 The ratio r_K

The ratio r_K is defined as

$$r_K \equiv \frac{f_+^{K^+\pi^+}(0)}{f_+^{K^0\pi^+}(0)} = (1.00027 \pm 0.00008) + (0.051 \pm 0.01)\,\varepsilon^{(2)}\ , \tag{33}$$

and parametrizes the isospin-breaking due to the initial kaon only, i.e. it is not sensitive to $\pi^0-\eta$ mixing. Compared to r, it cannot be expressed in simple analytic form, and is defined only for $q^2 = 0$.

The errors are estimated as twice the shift induced by varying the η mass between its physical and theoretical value, Eq.(23). Still, as discussed in the previous section, the sensitivity to the spectator quarks is negligible at $\mathcal{O}(p^4\varepsilon^{(2)})$, and the above errors probably underestimate the full $\mathcal{O}(p^6\varepsilon^{(2)})$ corrections. As before, a full analysis goes beyond our scope, and we consider only local contributions which are

$$(r_K)_{CT-\mathcal{O}(p^6\varepsilon^{(2)})} = -\varepsilon^{(2)}\frac{8}{\sqrt{3}}\delta_{CT} - 64\pi\alpha Z_{em}\frac{\Delta}{F_\pi^2}C_{12}\ . \tag{34}$$

	BR(%)	$\mathcal{I}_\ell$
K_{e3}^L	40.563(74)	0.15454(29)
$K_{\mu3}^L$	27.047(71)	0.10209(31)
K_{e3}^+	5.0758(290)	0.15889(30)
$K_{\mu3}^+$	3.3656(280)	0.10505(32)
K_{e3}^S	0.07046(91)	0.15454(29)

Table 1: Result of the best-fit of ISTRA [29], KLOE [30, 31], KTeV [32] and NA48 [33] data for the $K_{\ell3}$ branching-ratios, as performed in [8], and phase-space integrals for each mode.

We discard the $O\left(p^4\alpha\right)$ correction, which is about 10% of the strong one if $C_{12} \approx C_{34}$. Interestingly, the combination of counterterms δ_{CT} is exactly the one occurring in the isospin limit at $\mathcal{O}(p^6)$ for the form-factor at the origin, Eq.(25), but is now suppressed by an additional $\varepsilon^{(2)}$ factor. Using the numerical estimate $\delta_{CT} = -0.016 \pm 0.016$, with an inflated error to account for the discrepancy between lattice (and CKM unitarity) and model estimates as well as for the neglect of $\mathcal{O}(p^6\varepsilon^{(2)})$ loop contributions, and with $\varepsilon^{(2)}$ from Eq.(19), we get a very precise estimate for r_K:

$$r_K = (1.00027 \pm 0.00008) + (0.12 \pm 0.07)\,\varepsilon^{(2)} = 1.0015 \pm 0.0007\ , \tag{35}$$

which should thus include the bulk of $\mathcal{O}(p^6\varepsilon^{(2)})$ effects.

4 Numerical analysis

We start in the next subsection with the estimation of the FCNC form-factor slopes and rare K decay phase-space integrals. Then, in the following subsection, we estimate the FCNC form-factors at the origin and discuss the $K_{\ell3}$ experimental situation. These values are then used in the third subsection to get the rare K decay matrix-elements, and finally, the last subsection deals with the long-distance QED corrections for $K^+ \to \pi^+\nu\bar{\nu}$.

4.1 Slopes and phase-space integrals

The $K_{\ell3}$ form-factor slopes are determined from Dalitz plot analyses (corrected for long-distance QED corrections). To an excellent approximation, the K^+ and K_L slopes are identical (see Eq.(27)). A best-fit analysis of ISTRA [29], KLOE [30, 31], KTeV [32] and NA48 [33] data was performed recently in Ref. [8], with the result

$$\begin{array}{l}\lambda'_+ = (24.82 \pm 1.10)\cdot 10^{-3}\ ,\\ \lambda''_+ = (1.64 \pm 0.44)\cdot 10^{-3}\ ,\\ \lambda_0 = (13.38 \pm 1.19)\cdot 10^{-3}\ ,\end{array} \quad \rho = \left(\begin{array}{ccc} 1 & -0.95 & 0.32 \\ & 1 & -0.43 \\ & & 1 \end{array}\right) . \tag{36}$$

From these slopes, the $K_{\ell3}$ phase-space integrals are known to about 1/3%, see Table 1. It should be noted that about a third of these errors is due to the poor quality of the fit (i.e., to scale factors) [8], originating in the strong experimental disagreement between λ_0 measurements. This will hopefully disappear in the near future.

Let us relate these slopes to those of the FCNC form-factors. From Eq.(27), we know that they are equal (up to the electromagnetic effects, already included in Δ_{EM}). From Eqs.(17), one can immediately get the isospin-breaking correction relating the CC and FCNC form-factor slopes

$$\frac{\lambda_+^{FCNC}}{\lambda_+^{CC}} = 0.9986 \pm 0.0002\ . \tag{37}$$

This $\mathcal{O}\left(p^4\varepsilon^{(2)}\right)$ correction is very small, actually too small to really account for higher order effects. Indeed, the linear slopes only arise at $\mathcal{O}\left(p^4\right)$, where they are dominated by the L_9 counterterm, as for the pion

form-factor from which L_9 is fixed. Since L_9 is dominated by vector-meson exchanges, it is well-known that large $SU(3)$ corrections at $\mathcal{O}(p^6)$ are then needed to account for $m_\rho \neq m_{K^*}$. In our case, we of course do not try to fix L_9 from the pion vector form-factor, but simply take the slopes as extracted from $K_{\ell 3}$ data. We thus remain with the $\mathcal{O}(p^6\varepsilon^{(2)})$ corrections, which we estimate from the measured $K^{*+} - K^{*0}$ mass difference as

$$\frac{\lambda_+^{FCNC}}{\lambda_+^{CC}} \approx \frac{m^2_{K^{*+}(892)}}{m^2_{K^{*0}(892)}} \approx 0.990\ . \tag{38}$$

Therefore, to estimate the phase-space integrals relevant for rare K decays, we will rescale both the linear and quadratic slopes by

$$\frac{\lambda_+^{FCNC}}{\lambda_+^{CC}} = 0.990 \pm 0.005\ . \tag{39}$$

This theoretical uncertainty will be seen to have a smaller impact on the phase-space integrals than the experimental uncertainties on the slopes themselves.

For the scalar form-factor slope, the experimental information is less precise. Further, it is not so clear which resonance should play the dominant role as the presence of the κ pole may blur the picture. We therefore assign a slightly larger error to account for $\mathcal{O}(p^6\varepsilon^{(2)})$ effects and set

$$\frac{\lambda_0^{FCNC}}{\lambda_0^{CC}} = 0.99 \pm 0.01\ . \tag{40}$$

Nevertheless, this slope only matters for $K_L \to \pi^0\mu^+\mu^-$, and even there, the impact of this additional uncertainty on the final result will be very limited.

Using the experimental slopes extracted from $K_{\ell 3}$, Eq.(36), we find

$$\mathcal{I}_\nu^+ = 0.15269 \pm 0.00028 \pm 0.00007\ , \tag{41a}$$
$$\mathcal{I}_\nu^L = \mathcal{I}_e^{V,A} = 0.16043 \pm 0.00030 \pm 0.00008\ , \tag{41b}$$
$$\mathcal{I}_\mu^V = 0.03766 \pm 0.00010 \pm 0.00003\ , \tag{41c}$$
$$\mathcal{I}_\mu^A = 0.08624 \pm 0.00068 \pm 0.00009\ , \tag{41d}$$

where the first error comes from the experimental error on the slopes, and the second from the unknown $\mathcal{O}(p^6\varepsilon^{(2)})$ terms. The correlation between $\mathcal{I}_\mu^V$ and $\mathcal{I}_\mu^A$ turns out to be of about 0.1% and can be safely neglected in evaluating $\mathcal{B}\left(K_L \to \pi^0\mu^+\mu^-\right)_{\text{DCPV}}$. Currently, the above errors are dominated by the experimental error on λ'_+ (except $\mathcal{I}_\mu^A$ for which λ_0 dominates). As a result, a reduction of about 50% for both the λ'_+ and λ_0 experimental errors, leads to a $\sim 50\%$ reduction for all the phase-space integral errors.

4.2 Form-factors at the origin

The $K_{\ell 3}$ experimental branching ratios are given in Table 1, as obtained in the best-fit analysis of Ref. [8] (together with the $\pi\pi$, $\pi\pi\pi$ modes and the $K_{S,L}$ and K^+ lifetimes, $\tau_L = 51.173\,(200)$ ns, $\tau_S = 0.08958\,(5)$ ns and $\tau^\pm = 12.3840\,(193)$ ns). From these values, and after removing the long-distance QED corrections [8, 15], one finds

$$\left.\begin{array}{ll} K_{e3}^L: & |V_{us}\times f_+^{K^0\pi^+}(0)| = 0.21638\,(55) \\ K_{\mu 3}^L: & |V_{us}\times f_+^{K^0\pi^+}(0)| = 0.21678\,(69) \\ K_{e3}^S: & |V_{us}\times f_+^{K^0\pi^+}(0)| = 0.21554\,(142) \end{array}\right\} |V_{us}\times f_+^{K^0\pi^+}(0)|_{\exp} = 0.21645\,(41)\ , \tag{42a}$$

$$\left.\begin{array}{ll} K_{e3}^+: & |V_{us}\times f_+^{K^+\pi^0}(0)| = 0.22248\,(73) \\ K_{\mu 3}^+: & |V_{us}\times f_+^{K^+\pi^0}(0)| = 0.22314\,(106) \end{array}\right\} |V_{us}\times f_+^{K^+\pi^0}(0)|_{\exp} = 0.22269\,(60)\ . \tag{42b}$$

The most precise strategy to get the FCNC form-factors at the origin from these data is to use the ratios r and r_K as

$$|V_{us} \times f_+^{K^0\pi^0}(0)| = r\, r_K \frac{|V_{us} \times f_+^{K^0\pi^+}(0)|^2_{\rm exp}}{|V_{us} \times f_+^{K^+\pi^0}(0)|_{\rm exp}} = 0.2107 \pm 0.0010\,, \tag{43}$$

$$|V_{us} \times f_+^{K^+\pi^+}(0)| = r_K\, |V_{us} \times f_+^{K^0\pi^+}(0)|_{\rm exp} = 0.2168 \pm 0.0004\,. \tag{44}$$

In this way, $\mathcal{O}(p^6\varepsilon^{(2)})$ corrections are under good theoretical control. Further, since the error on $r_K\,(r)$ accounts only for $\sim 10\%\,(3\%)$ and $\sim 28\%\,(0\%)$ of the final error, there is much room for improvements on the experimental side.

Discussion: The strategy of using r instead of r_{0+} [4] to get $|V_{us} \times f_+^{K^0\pi^0}(0)|$ is more precise since, as discussed previously, higher order effects are under better control for r (see Eq.(29) and (31)). On the other hand, the disadvantage of using r is that we directly rely on $K_{\ell 3}^+$ data to estimate isospin-breaking effects.

In this respect, it should be noted that neutral $K_{\ell 3}^0$ data are in much better shape than the charged $K_{\ell 3}^+$ data. Indeed, all the new generation Kaon experiments (designed for $K^0 - \bar{K}^0$ mixing) have published the whole set of information relevant for neutral modes, i.e. branching ratios, K_L lifetime and phase-space measurements. In this way, $V_{us} \times f_+^{K^0\pi^+}(0)$ and its slopes are known with very high accuracy. At the moment this situation is not shared by the charged modes, for which the analysis relies on much older data, and for which the treatment of radiative corrections is unclear. New preliminary measurements have been announced by NA48 and ISTRA+ for the ratios $\mathcal{B}(K_{\ell 3}^+)/\mathcal{B}\left(K^+ \to \pi^+\pi^0\right)$, while KLOE has announced preliminary results for the absolute branching ratios [31]

$$\mathcal{B}(K_{e3}^+) = 4.965\,(52)\,\%,\;\; \mathcal{B}(K_{\mu 3}^+) = 3.233\,(39)\,\%\,. \tag{45}$$

All these preliminary values are included in the best-fit analysis of Ref. [8] (Table 1). Unfortunately, the old PDG average for $\mathcal{B}\left(K^+ \to \pi^+\pi^0\right)$ is rather suspicious (see the discussion in Ref. [8])), while KLOE data are not precise enough to really compete yet. As a result, the fit to $K_{\ell 3}^+$ data of Ref. [8] has a rather bad χ^2. This situation should soon improve, as both $K^+ \to \pi^+\pi^0$ and $K_{\ell 3}^+$ modes are currently under study.

The above comment can be made more quantitative by looking at the evolution of the experimental determination of the r_{0+} ratio as new $K_{\ell 3}^+$ data are announced:

$$(r_{0+})_{\rm exp} = \frac{|V_{us} \times f_+^{K^+\pi^0}(0)|_{\rm exp}}{|V_{us} \times f_+^{K^0\pi^+}(0)|_{\rm exp}} = \left\{\begin{array}{ll} 1.0328 \pm 0.0039 & \text{Average } K_{\ell 3}^+ \text{ data, before Kaon 2007 [8],} \\ 1.0288 \pm 0.0034 & \text{Average } K_{\ell 3}^+ \text{ data, after Kaon 2007 [8],} \\ 1.0140 \pm 0.0046 & \text{KLOE preliminary } K_{\ell 3}^+ \text{ data alone [31].} \end{array}\right. \tag{46}$$

Within 2σ, these determinations agree with r_{0+} estimated at $\mathcal{O}\left(p^4\varepsilon^{(2)}\right)$ from the $\pi^0 - \eta$ mixing parameters, Eqs.(19–21),

$$(r_{0+})_{\rm th} = 1 + \sqrt{3}(\varepsilon^{(2)} + \varepsilon^{(4)}) = 1.0238 \pm 0.0022\,, \tag{47}$$

where errors are added quadratically. The $\mathcal{O}(p^6\varepsilon^{(2)})$ corrections (see Eq.(29)) are accounted for in the above error, assuming they scale as $\varepsilon^{(6)}/\varepsilon^{(4)} \sim \varepsilon^{(4)}/\varepsilon^{(2)} \sim 30\%$[1]. Therefore, waiting for the $K_{\ell 3}^+$ situation to settle, we will also present final estimates for the rare K decay branching ratios based on the theoretical r_{0+}, Eq.(47), together with $K_{\ell 3}^0$ data only, corresponding to

$$|V_{us} \times f_+^{K^0\pi^0}(0)|_{\rm th} = \frac{r\, r_K}{(r_{0+})_{\rm th}} |V_{us} \times f_+^{K^0\pi^+}(0)|_{\rm exp} = 0.2118 \pm 0.0004_{\rm exp} \pm 0.0005_{\rm th}\,. \tag{48}$$

Note that in principle, if Eq.(47) is assumed to hold, one should perform the best-fit average of $K_{\ell 3}^0$ and $K_{\ell 3}^+$ data to get $|V_{us} \times f_+^{K^0\pi^+}(0)|_{\rm av.} = 0.21669\,(36)$, and then use this value to estimate both FCNC form-factors. As this has only a very small impact numerically, and since we argued that the recent neutral $K_{\ell 3}^0$ data are in much better shape than the older K^+ data, we prefer not to use this average.

[1] Comparing the local $\mathcal{O}(p^6)$ terms δ_{CT}, Eq.(25), to those for $\varepsilon^{(6)}$ as well as for the spectator quark effects, Eq.(29), one can note the larger numerical coefficients for the latter. Also, note that the spectator quark effects in r_{0+} start only at $\mathcal{O}(p^6\varepsilon^{(2)})$. For these reasons, the naive counting could have to be slightly amended.

	$(r_{0+})_{\rm th}$	$(r_{0+})_{\rm exp}$	τ_+	$f(0)_{K_{\ell 3}}$	$\mathcal{I}$	r_K	r	Future ?
κ_ν^+	0.5173 ± 0.0025	0.5173 ± 0.0025	19	43	21	17	–	± 0.0023
κ_ν^L	2.231 ± 0.013	2.209 ± 0.017	–	76	12	10	2	± 0.012
κ_e^V	0.7832 ± 0.0044	0.7755 ± 0.0058	–	76	12	10	2	± 0.0044
κ_μ^V	0.1838 ± 0.0011	0.1821 ± 0.0014	–	71	18	9	2	± 0.0010
κ_μ^A	0.4210 ± 0.0040	0.4169 ± 0.0045	–	54	37	7	2	± 0.0029

Table 2: Final results for the rare K decay rate coefficients, in units of $10^{-10}(|V_{us}|/0.225)^8$, for the r_{0+} estimates of Eq.(47) and (46) (current average), respectively. For the latter, the approximate breakdown of the errors (in %) is also indicated. The last column shows the possible improvements achievable by a 50% reduction in the $f_+^{K^+\pi^0}(0)$, λ'_+ and λ_0 experimental errors.

4.3 Coefficients for the rare K decay branching ratios

For the κ coefficients entering the rare K branching ratios, Eqs.(10) and (13), one can in principle avoid propagating the error due to the K_L, K^+ lifetimes by using, instead of Eqs.(42), the averages:

$$(\tau_L|V_{us}\times f_+^{K^0\pi^+}(0)\,|^2)_{\rm exp} = 0.23978\,(64)\cdot 10^{-8}\ , \tag{49a}$$

$$(\tau_+|V_{us}\times f_+^{K^+\pi^0}(0)\,|^2)_{\rm exp} = 0.6141\,(32)\cdot 10^{-9}\ . \tag{49b}$$

Importantly, we do not use directly the $K_{\ell 3}$ branching ratios, as is usually done following Ref. [4], because the radiative correction factors $\Delta^{ij}_{\ell,EM}$ have to be removed first (see Eq.(8)).

From these averages, the K_L decay coefficients can be expressed as

$$\kappa_\nu^L = \frac{G_F^2 M_K^5 \alpha\,(M_Z)^2}{256\pi^5\sin^4\theta_W}\,|V_{us}|^8\ \ \tau_L\ \left(r\,r_K\frac{|V_{us}\times f_+^{K^0\pi^+}(0)\,|^2_{\rm exp}}{|V_{us}\times f_+^{K^+\pi^0}(0)\,|_{\rm exp}}\right)^2\ \mathcal{I}_\nu^0 \tag{50}$$

$$= \frac{G_F^2 M_K^5 \alpha\,(M_Z)^2}{256\pi^5\sin^4\theta_W}\,|V_{us}|^8\ \ (\tau_L|V_{us}\times f_+^{K^0\pi^+}(0)\,|^2)_{\rm exp}\ \left(\frac{r\,r_K}{r_{0+}}\right)^2\ \mathcal{I}_\nu^0\ , \tag{51}$$

and similarly for $\kappa_{e,\mu}^{V,A}$. The second form gives the smallest errors, i.e. optimizes the use of the experimental information. On the other hand, doing the same for the K^+ decay would increase the error because r_{0+} has a larger impact than τ_+:

$$\kappa_\nu^+ = \frac{G_F^2 M_K^5 \alpha\,(M_Z)^2}{256\pi^5\sin^4\theta_W}\,|V_{us}|^8\ \ \tau_+\ (r_K|V_{us}\times f_+^{K^0\pi^+}(0)\,|_{\rm exp})^2\ \mathcal{I}_\nu^+ \tag{52}$$

$$= \frac{G_F^2 M_K^5 \alpha\,(M_Z)^2}{256\pi^5\sin^4\theta_W}\,|V_{us}|^8\ \ (\tau_+|V_{us}\times f_+^{K^+\pi^0}(0)\,|^2)_{\rm exp}\ \left(\frac{r_K}{r_{0+}}\right)^2\ \mathcal{I}_\nu^+\ , \tag{53}$$

Numerically, the first form leads to smaller errors.

Using in addition the values quoted in Eq.(11) for $\alpha\,(M_Z)$ and $\sin^2\theta_W$, we get the results given in Table 2, for the experimental or theoretical r_{0+} estimates discussed in the previous Section[2]. The breakdown of the error (in percent) is given for the $(r_{0+})_{\rm exp}$-based estimates, and shows that the experimental errors (on the K^+ lifetime τ_+, on the $K_{\ell 3}$ form-factors at the origin $f(0)_{K_{\ell 3}}$ and on the phase-space integrals $\mathcal{I}$) completely dominate, so that there is much room for improvements. In particular, the last column shows the errors one would get if the errors on $|V_{us}\times f_+^{K^+\pi^0}(0)\,|_{\rm exp}$ (from $K_{\ell 3}^+$, currently the most limiting), λ'_+ and λ_0 were all reduced by a factor of two.

[2]It should be noted that if the experimental uncertainty on $\sin^2\theta_W$ was to be included in the κ_ν^+ and κ_ν^L coefficients, their errors would increase by 14% and 11%, respectively. Since the present work concerns isospin-breaking effects, these uncertainties are left aside. Further, as said in the first section, they should be dealt with, together with higher-order electroweak effects, at the level of the short-distance Wilson coefficients [11].

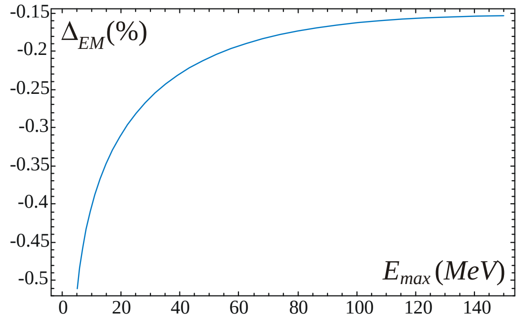

Figure 1: The QED correction to $K^+ \to \pi^+\nu\bar{\nu}\,(\gamma)$, in %, as a function of the maximum energy of the undetected photon.

Finally, of special interest for the search of new physics, the ratio of the two neutrino modes can be predicted with good accuracy:

$$\frac{\kappa_\nu^+}{\kappa_\nu^L} = \frac{1}{r^2}\,\frac{M_{K^+}^5\,\mathcal{I}_\nu^+\,(\tau_+|V_{us}\times f_+^{K^+\pi^0}\,(0)\,|^2)_{\text{exp}}}{M_{K^0}^5\,\mathcal{I}_\nu^0\,(\tau_L|V_{us}\times f_+^{K^0\pi^+}\,(0)\,|^2)_{\text{exp}}} = 0.2359 \pm 0.0015\;. \tag{54}$$

Further, the error is currently dominated by $(\tau_+|V_{us}\times f_+^{K^+\pi^0}\,(0)\,|^2)_{\text{exp}}$ and could easily be reduced by a factor of two in the near future.

4.4 QED radiative corrections

The final piece needed is the long-distance QED correction to $K^+ \to \pi^+\nu\bar{\nu}$. For that, we have to combine the virtual photon correction given in Eq.(17d) with the real photon emissions. These have to be computed at $\mathcal{O}(p^2\alpha)$, leading to the radiative decay rate (see Appendix C for details and explicit expressions):

$$\Gamma\left(K^+ \to \pi^+\nu\bar{\nu}\gamma\right) = \frac{G_F^2 M_K^5 \alpha\,(M_Z)^2\,|y_\nu|^2}{256\pi^5\sin^4\theta_W}\,\frac{\alpha}{\pi}\int_0^{(1-r_\pi)^2} dz\lambda_\pi^3 J_{BR}\,(z,r_\pi,E_{\max})\;. \tag{55}$$

The function $J_{BR}\,(z,r_\pi,E_{\max})$ accounts for photon emission with energies up to $E_{\max}$. Altogether, the IR-finite long-distance QED correction is

$$\Delta_{EM}\,(E_{\max}) = \frac{\alpha}{\pi}\frac{1}{\mathcal{I}_\nu^+}\int_0^{(1-r_\pi)^2} dz\lambda_\pi^3\,(J_{BR}\,(z,r_\pi,E_{\max}) - J_{EM}\,(z,r_\pi))\;. \tag{56}$$

As shown in Fig.1, $\Delta_{EM}\,(E_{\max})$ is of less than one percent for reasonable $E_{\max}$, while the fully inclusive correction is $\Delta_{EM} = -0.15\%$, and is thus comparable to those of $K_{\ell 3}$ in magnitude. Still, being free of counterterms at leading order, it is precisely determined. Looking at Table 2, it is of the order of the current uncertainty on κ_ν^+.

5 Conclusion

In this paper, we have presented a theoretical strategy to extract rare K decay matrix-elements from $K_{\ell 3}$ data to a few parts per mil. In summary, from the measured $K_{\ell 3}$ branching ratios and form-factor slopes:

Step 1: The $K_{\ell 3}$ phase-space integrals are directly obtained from the measured slopes, see Eq.(15) and Appendix A.

Step 2: The $K_{\ell 3}$ form-factors at the origin (modulo V_{us}, kept as a parametric uncertainty) are obtained after removing the QED corrections and the phase-space dependences from the $K_{\ell 3}^{L,S,+}$ branching ratios, and are separately averaged for the charged and neutral $K_{\ell 3}$ decays, see Eqs.(42) and (49).

The outcome of these first two steps is provided by experimentalists, as in Ref. [8], taking into account the correlations among slopes, branching ratios and QED corrections. Then:

Step 3: The slopes are rescaled to account for isospin-breaking, Eqs.(39,40), and rare K decay phase-space integrals are computed (again, taking care of the correlations among slopes), see Eqs.(16) and Appendix A.

Step 4: Using the averaged values for the neutral and charged $K_{\ell 3}$ form-factors at the origin together with the ratios r and r_K, Eqs.(32) and (35), the rare K decay form-factors at the origin or, even better, directly the coefficients κ are obtained through Eqs.(51,52).

Using this strategy, exceptional control over the hadronic physics is possible thanks to the two ratios of form-factors r and r_K, for which higher-order corrections happen to be very suppressed. Overall, this leads to a significant improvement over the leading-order analysis of Ref. [4]. In addition, for the first time, uncertainties due to higher-order effects in the chiral expansion were carefully studied, allowing us to carry a thorough error analysis. Finally, QED corrections are accounted for, both for $K_{\ell 3}$ and rare K decays.

The final results for the coefficients entering the rare K decay rates, Eqs.(10,13), are collected in Table 2. As explained in the text, there is at present an open question related to the isospin-breaking effects observed in the ratio of the $K_{\ell 3}^{+}$ form-factor over the $K_{\ell 3}^{0}$ one (see Eq.(46)). The current best-fit $K_{\ell 3}^{+}$ branching ratios relies on a number of quite old measurements and has a poor χ^2 [8]. This problem will hopefully disappear in the near future, as the K^+ modes are currently under experimental study. Therefore, waiting for the $K_{\ell 3}^{+}$ situation to settle, we consider the values in the first column of Table 2 as our best estimates.

Compared to the matrix-elements used in Ref. [1], based on the leading order analysis of Ref. [4], the error for κ_ν^+ $\left(\kappa_\nu^L\right)$ is reduced by a factor of about $7\,(4)$, respectively. Using these new values, together with the Wilson coefficient Eq.(4) and the CKM parameters Eq.(6), we find (adding errors quadratically)

$$\begin{array}{ll}\mathcal{B}\left(K_L \to \pi^0 \nu\bar{\nu}\right) = (2.49 \pm 0.39) \times 10^{-11} & \left(X_t : 25\%,\ \kappa_\nu^L : 3\%,\ \text{CKM} : 72\%\right) \\ \mathcal{B}\left(K^+ \to \pi^+ \nu\bar{\nu}\,(\gamma)\right) = (7.83 \pm 0.82) \times 10^{-11} & \left(X_t : 20\%,\ P_{u,c} : 31\%,\ \kappa_\nu^+ : 3\%,\ \text{CKM} : 46\%\right)\end{array} \tag{57}$$

where we have taken $\Delta_{EM}\,(E_{\max}^{\gamma} \approx 20\,\text{MeV}) \approx -0.30\%$. As the breakdown of the error shows, the long-distance uncertainties due to the matrix-elements are now negligible compared to the other (non-parametric) theoretical uncertainties. Similarly, for the direct CP-violating contribution to $K_L \to \pi^0 \ell^+ \ell^-$, the (non-parametric) error is now completely dominated by short-distance uncertainties in the Wilson coefficients y_{7A} and y_{7V}, Eq.(5). However, for these modes, it is the uncertainty over the indirect CP-violating contribution which is at present the most limiting, and for that, better measurements of $\mathcal{B}\left(K_S \to \pi^0 \ell^+ \ell^-\right)$ would be necessary. The impact of our new values on the $\mathcal{B}\left(K_L \to \pi^0 \ell^+ \ell^-\right)$ predictions is therefore limited and we refer to [3] for the numerics.

In conclusion, the hadronic uncertainties due to the matrix-elements of dimension-six FCNC operators are now well below the short-distance uncertainties in $K \to \pi\nu\bar{\nu}$ and $\mathcal{B}\left(K_L \to \pi^0 \ell^+ \ell^-\right)_{\text{DCPV}}$. In addition, being dominated by $K_{\ell 3}$ experimental errors, they should become even more accurate in the near future since it is reasonable to expect slope measurements and $K_{\ell 3}^{+}$ data to further improve. Hadronic uncertainties will thus not hamper our ability to test the Standard Model with very high precision.

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A Phase-space integral coefficients

Expanding the phase-space integrals in powers of the slopes, and carrying out the integration, they can be expressed as numerical polynomials

$$\mathcal{I} = c_0 + c_1\lambda'_+ + c_2\left(\lambda'^2_+ + \lambda''_+\right) + c_3\lambda''^2_+ + c_4\lambda'_+\lambda''_+ + c_5\lambda_0 + c_6\lambda_0^2\,, \tag{58}$$

with the coefficients for each of the nine phase-space integrals collected in the table below:

	c_0	c_1	c_2	c_3	c_4	c_5	c_6
$\mathcal{I}_e^{0+}$	0.14084	0.4867	0.672	2.276	2.318	–	–
$\mathcal{I}_\mu^{0+}$	0.09085	0.2938	0.474	1.784	1.751	0.2055	0.360
$\mathcal{I}_e^{+0}$	0.14480	0.5006	0.693	2.354	2.392	–	–
$\mathcal{I}_\mu^{+0}$	0.09346	0.3023	0.489	1.846	1.809	0.2120	0.373
$\mathcal{I}_\nu^{+}$	0.13958	0.4721	0.638	2.068	2.153	–	–
$\mathcal{I}_\nu^{0}, \mathcal{I}_e^{V,A}$	0.14603	0.5156	0.729	2.586	2.572	–	–
$\mathcal{I}_\mu^{V}$	0.03092	0.2299	0.453	2.084	1.892	–	–
$\mathcal{I}_\mu^{A}$	0.07665	0.0691	0.150	0.804	0.681	0.5526	1.198

B Loop functions

Explicit representations for the loop functions in the momentum range $0 < q^2 < (M_{K^i} - M_{\pi^j})^2$ can be written in terms of $x = q^2/M_{K^i}^2, r_\pi = M_{\pi^j}/M_{K^i}$, $r_\gamma = M_\gamma/M_{K^i}$ and

$$z_{1,2} = 1 - r_\pi^2 \mp x,\ \ \beta = 1 + r_\pi^2 - x,\ \ \lambda_\pi = \sqrt{r_\pi^4 - 2\left(1+x\right)r_\pi^2 + \left(1-x\right)^2}\,, \tag{59}$$

as

$$\begin{aligned} H_i^j\left(x, r_\pi\right) = \frac{M_i^2}{24\left(4\pi F_\pi\right)^2}\Bigg(&\frac{z_2\lambda_\pi^2 + 2(r_\pi^2-(1-x)^2)x}{x^2}\log r_\pi - \frac{\lambda_\pi^3}{2x^2}\log\frac{\beta-\lambda_\pi}{\beta+\lambda_\pi} + \frac{\left(1-r_\pi^2\right)^2}{x} \\ &-4\left(1+r_\pi^2\right) + x\left(128\pi^2 L_9^r - \log\frac{M_{K^i}^2}{\mu^2} + \frac{5}{3}\right)\Bigg)\,, \end{aligned} \tag{60}$$

and (with $x = q^2/M_{K^\pm}^2$ and $r_\pi = M_{\pi^\pm}/M_{K^\pm}$)

$$J_{EM}\left(x, r_\pi\right) = \left(2 + 2\log r_\gamma - \log r_\pi\right)\left(1 + \frac{\beta}{2\lambda_\pi}\log\left(\frac{\beta-\lambda_\pi}{\beta+\lambda_\pi}\right)\right) + \frac{\beta}{4\lambda_\pi}C\left(x, r_\pi\right)\,, \tag{61}$$

$$\begin{aligned} C\left(x, r_\pi\right) = {}&\log^2\left(\frac{z_1+\lambda_\pi}{2x}\right) - \log^2\left(\frac{z_1-\lambda_\pi}{2x}\right) + \log^2\left(\frac{z_2-\lambda_\pi}{2x}\right) - \log^2\left(\frac{z_2+\lambda_\pi}{2x}\right) \\ &+ 2\log\left(\frac{z_1-\lambda_\pi}{z_1+\lambda_\pi}\right)\log\left(\frac{z_1+\lambda_\pi}{2\lambda_\pi}\right) + 2\log\left(\frac{2\lambda_\pi}{z_2-\lambda_\pi}\right)\log\left(\frac{z_2+\lambda_\pi}{2\lambda_\pi}\right) \\ &- \log\frac{\lambda_\pi^2}{r_\pi^2}\log\left(\frac{\beta-\lambda_\pi}{\beta+\lambda_\pi}\right) - 4\,\mathrm{Li}_2\left(\frac{\lambda_\pi - z_2}{2\lambda_\pi}\right) - 4\,\mathrm{Li}_2\left(\frac{z_1-\lambda_\pi}{z_1+\lambda_\pi}\right)\,. \end{aligned} \tag{62}$$

The IR-finite remainder obeys $C\left(0, r_\pi\right) = 0$. Also, one can check that $J_{EM}\left(0,1\right) = 0$, as required by vector-current conservation.

C Bremsstrahlung and QED correction

For the bremsstrahlung process $K^+\left(P\right) \to \pi^+\left(K\right)\nu\left(p_1\right)\bar{\nu}\left(p_2\right)\gamma\left(k\right)$, the $\mathcal{O}\left(p^2 e\right)$ amplitude is given by

$$\mathcal{M}^\mu = \frac{G_F}{\sqrt{2}}\frac{2\,e\,\alpha\left(M_Z\right)\,y_\nu}{2\pi\sin^2\theta_W}\left(g^{\mu\alpha} - \frac{2K^\alpha P^\mu}{M_\gamma^2 + 2K\cdot k} - \frac{2P^\alpha K^\mu}{M_\gamma^2 - 2P\cdot k}\right)\bar{u}(p_\nu)\gamma_\mu\left(1-\gamma^5\right)v\left(p_{\bar{\nu}}\right)\,, \tag{63}$$

since the vector form-factor is $f_+ = 1$ at this order. Keeping $M_\gamma > 0$, this amplitude has to be squared and integrated over the four-body phase-space, decomposed as

$$d\Phi_4\left(P;K,k,p_\nu,p_{\bar\nu}\right) = \frac{dT^2}{2\pi} d\Phi_3\left(P;K,T,k\right) d\Phi_2\left(T;p_\nu,p_{\bar\nu}\right) \ . \tag{64}$$

The two-body leptonic integral can be done immediately, while the two variables for the three-body integral are chosen as the photon and pion energies. Introducing the reduced variables $x = T^2/M_K^2$, $z = 2E_\pi/M_K$ and $\omega = 2E_\gamma/M_K$, the differential rate is

$$\frac{d\Gamma\left(x,\omega,z\right)}{dxd\omega dz} = \frac{G_F^2 M_K^5 \alpha\left(M_Z\right)^2 \left|y_\nu\right|^2}{256\pi^5 \sin^4\theta_W} \frac{\alpha}{\pi} \left(\frac{zz_2}{\omega} - \frac{\lambda_\pi^2}{\omega^2} - \frac{z+\omega}{\omega} \frac{\left(z_1-\omega\right)\left(2-z-\omega\right) - 2xz}{\beta - z - \omega - r_\gamma^2} \right.$$
$$\left. - r_\pi^2 \frac{\left(2-z-\omega\right)^2 - 4x}{\left(\beta - z - \omega - r_\gamma^2\right)^2} + 2z_2 - z - \omega + r_\pi^2 \right) \ , \tag{65}$$

where terms leading to $\mathcal{O}(r_\gamma)$ contributions to the rate have been neglected. The integration bounds are

$$0 \le x \le \left(1-r_\pi\right)^2 \ , \ \ 2r_\gamma \le \omega \le 1 + r_\gamma^2 - \left(r_\pi + \sqrt{x}\right)^2 \ , \ \ a\left(x,y\right) - b\left(x,y\right) \le z \le a\left(x,y\right) + b\left(x,y\right) \ , \tag{66}$$
$$a\left(x,y\right) = \frac{\left(2-\omega\right)\left(\beta + r_\gamma^2 - \omega\right)}{2\left(1+r_\gamma^2-\omega\right)}, b\left(x,y\right) = \frac{\sqrt{\omega^2 - 4r_\gamma^2}\sqrt{\left(z_1 - \omega + r_\gamma^2\right)^2 - 4r_\pi^2 x}}{2\left(1+r_\gamma^2-\omega\right)} \ .$$

There is no need to cut the x integral since the differential rate with respect to x is, in principle, an observable and is therefore IR-safe.

To account for the experimental constraints, the photon integral will be limited to $E_{\max}$, i.e. $\omega_{\max} = 2E_{\max}/M_K$. Since there will always be some x for which $1 + r_\gamma^2 - \left(r_\pi + \sqrt{x}\right)^2 < \omega_{\max}$, the phase-space boundaries become quite involved. To simplify them, we make use of the fact that the probability amplitude becomes imaginary outside the physical phase-space and write

$$\int_0^{\left(1-r_\pi\right)^2} dx \int_{2r_\gamma}^{1+r_\gamma^2-\left(r_\pi+\sqrt{x}\right)^2} d\omega\, \theta\left(\omega_{\max} - \omega\right) = \mathrm{Re}\int_0^{\left(1-r_\pi\right)^2} dx \int_{2r_\gamma}^{\omega_{\max}} d\omega \ , \tag{67}$$

as can be explicitly checked numerically. When working to leading order in $\omega_{\max}$, the spurious imaginary part cancels out and after integration over z and ω, we find the differential rate:

$$\Gamma\left(K^+ \to \pi^+ \nu\bar\nu\gamma\right) = \frac{G_F^2 M_K^5 \alpha\left(M_Z\right)^2 \left|y_\nu\right|^2}{256\pi^5 \sin^4\theta_W} \frac{\alpha}{\pi} \int_0^{\left(1-r_\pi\right)^2} dx \lambda_\pi^3 J_{BR}\left(x, r_\pi, E_{\max}\right) \ , \tag{68}$$

where

$$J_{BR}\left(x,r_\pi,E_{\max}\right) = -\left(2 + \frac{\beta}{\lambda}\log\frac{\beta-\lambda_\pi}{\beta+\lambda_\pi}\right)\left(\log\frac{\omega_{\max}}{r_\gamma} - \frac{1}{4}\log\left(\frac{\beta-\lambda_\pi}{\beta+\lambda_\pi}\right) - \frac{1}{2} - \frac{\omega_{\max}}{\beta}\right)$$
$$- \left(\frac{1}{2} + \frac{\beta}{\lambda_\pi}\right)\log\left(\frac{\beta-\lambda_\pi}{\beta+\lambda_\pi}\right) + \frac{\beta}{\lambda}\,\mathrm{Li}_2\left(\frac{2\lambda_\pi}{\lambda_\pi - \beta}\right) - \frac{2\omega_{\max}}{\beta} + \mathcal{O}\left(\omega_{\max}\right) \ . \tag{69}$$

As can be checked, combining this with virtual photon corrections, the IR-divergence indeed cancels out.

For numerical applications, we use the following numerical interpolation for the full correction (virtual + real photons)

$$J_{BR}\left(x,r_\pi,E_{\max}\right) - J_{EM}\left(x,r_\pi\right) = j_0\left(\omega_{\max}\right)\ \left(1 + x\, j_1\left(\omega_{\max}\right) + \mathcal{O}\left(x^2\right)\right) \ , \tag{70}$$
$$j_0\left(a\right) = 0.595 + 0.965\log a - 1.730a + 0.201a^2 + 0.114a^3 \ ,$$
$$j_1\left(a\right) = -2.129 - 0.0711\log a + 0.496a + 5.977a^2 - 5.585a^3 \ .$$

It is valid to better than 1% for all allowed values of $\omega_{\max}$. When integrated over x, it gives an approximation for the total QED contribution to the inclusive rate to within 5%, more than sufficient since the QED correction itself is very small compared to isospin-breaking corrections.

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